

MAS212 Linear Algebra I Exercises 4 (with solutions)

© Francis J. Wright, 2006

Assessed submission deadline 5pm on Friday 27th October 2006 in the blue box on the ground floor. Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together. Marks out of 100 are shown in square brackets after each question number.

1. [20] Find basis sets for the following vector subspaces of $\mathbb{R}^3$ and hence determine their dimensions:

- (a) $\{(x, y, z) \mid x = 2y\}$;
- (b) $\{(x, y, z) \mid x = -y, z = 0\}$.

Solution

NB: Basis sets are not unique but dimension is unique!

(a) The vector subspace can be written parametrically as

$$\{(2y, y, z) \mid y, z \in \mathbb{R}\} = \{y(2, 1, 0) + z(0, 0, 1) \mid y, z \in \mathbb{R}\}.$$

Hence $\{(2, 1, 0), (0, 0, 1)\}$ is a spanning set, which is also obviously linearly independent since one vector is not a scalar multiple of the other and so is a basis set. Therefore the dimension is 2.

(b) The vector subspace can be written parametrically as

$$\{(x, -x, 0) \mid x \in \mathbb{R}\} = \{x(1, -1, 0) \mid x \in \mathbb{R}\}.$$

Hence $\{(1, -1, 0)\}$ is a spanning set, which is also obviously linearly independent since it consists of one nonzero vector and so is a basis set.

Therefore the dimension is 1.

2. [20] Find basis sets for the following vector subspaces of $\mathbb{R}^4$ and hence determine their dimensions:

- (a) $\{(w, x, y, z) \mid w - x + y = 0\}$;
- (b) $\{(w, x, y, z) \mid 2w - x = 0, x + y = 0, y - 2z = 0\}$.

Solution

(a) The constraint can be solved for w (say) to give $w = x - y$, so the subspace can be written parametrically as

$$\{(x - y, x, y, z) \mid x, y, z \in \mathbb{R}\} = \{x(1, 1, 0, 0) + y(-1, 0, 1, 0) + z(0, 0, 0, 1) \mid x, y, z \in \mathbb{R}\}.$$

$$\{(1, 1, 0, 0), (-1, 0, 1, 0), (0, 0, 0, 1)\} \text{ is a spanning set.}$$
$$x(1, 1, 0, 0) + y(-1, 0, 1, 0) + z(0, 0, 0, 1) = 0 \Rightarrow x - y = 0, x = 0, y = 0, z = 0 \Rightarrow x = 0, y = 0, z = 0.$$

Hence the set of vectors is linearly independent and so is a basis set. Therefore the dimension is 3.

(b) The constraints can be written as (say) $x = 2w, y = -x = -2w, z = \frac{y}{2} = -w$, so the subspace can

be written parametrically as $\{(w, 2w, -2w, -w) \mid w \in \mathbb{R}\} = \{w(1, 2, -2, -1) \mid w \in \mathbb{R}\}$. Hence $\{(1, 2, -2, -1)\}$ is a spanning set, which is also obviously linearly independent since it consists of one nonzero vector and so is a basis set. Therefore the dimension is 1.

3. [30] Let U and W be vector subspaces of $\mathbb{R}^4$ defined by

$$U = \{(x, y, z, t) \mid x = t\}, \quad W = \{(x, y, z, t) \mid y = z\}.$$

Find bases for U , W and $U \cap W$. Then use the dimension theorem

$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$ to find the dimension of the subspace $U + W$ and confirm the result by constructing a basis for $U + W$ from the bases for U , W . [You are not required to prove any claims about basis sets.]

Solution

$U = \{(x, y, z, x) \mid x, y, z \in \mathbb{R}\} = \{x(1, 0, 0, 1) + y(0, 1, 0, 0) + z(0, 0, 1, 0) \mid x, y, z \in \mathbb{R}\}$ so $\{(1, 0, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0)\}$ is a spanning set for U . It is obviously linearly independent so it is also a basis for U and $\dim(U) = 3$.

$W = \{(x, y, y, t) \mid x, y, t \in \mathbb{R}\} = \{x(1, 0, 0, 0) + y(0, 1, 1, 0) + t(0, 0, 0, 1) \mid x, y, t \in \mathbb{R}\}$ so $\{(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$ is a basis for W and $\dim(W) = 3$.

$U \cap W = \{(x, y, y, x) \mid x, y \in \mathbb{R}\} = \{x(1, 0, 0, 1) + y(0, 1, 1, 0) \mid x, y \in \mathbb{R}\}$ so $\{(1, 0, 0, 1), (0, 1, 1, 0)\}$ is a basis for $U \cap W$ and $\dim(U \cap W) = 2$.

Hence, $\dim(U + W) = 3 + 3 - 2 = 4$. (Therefore, $U + W = \mathbb{R}^4$.)

$U + W$ is spanned by the union of the bases for U and W , namely

$\{(1, 0, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0), (1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$, from which we must construct a maximal linearly independent set. We can re-order this set as $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, 0, 1), (0, 1, 1, 0)\}$. The last 2 vectors are obvious linear combinations of the first 4 vectors, which are the standard basis vectors for $\mathbb{R}^4$.

Therefore $U + W = \mathbb{R}^4$ and $\dim(U + W) = 4$.

4. [30] V is the real vector space consisting of 2×2 matrices with real entries together with the usual rules for addition and scalar multiplication. U is the subset consisting of all matrices of the

form $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$, $a, b, c \in \mathbb{R}$, and W is the subset consisting of all matrices of the form

$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, $a, b, c \in \mathbb{R}$. Prove that U is a vector subspace of V , hence deduce that W is also a vector

subspace of V , and find a basis for each of U and W . Find a basis for $U \cap W$ and a basis for $U + W$, and hence verify that these vector subspaces satisfy the dimension theorem given in Question 3.

[You are not required to prove any claims about basis sets.]

Solution

Proof that U is a vector subspace of V : Setting $a = b = c = 0 \in \mathbb{R}$ gives the zero matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in U, \text{ which is the zero vector in } U.$$

Let $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$ so that $\begin{bmatrix} a_1 & b_1 \\ c_1 & a_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & a_2 \end{bmatrix} \in U$. Then for all $k_1, k_2 \in \mathbb{R}$,

$$k_1 \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & a_1 \end{bmatrix} + k_2 \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & a_2 \end{bmatrix} = \begin{bmatrix} k_1 a_1 + k_2 a_2 & k_1 b_1 + k_2 b_2 \\ k_1 c_1 + k_2 c_2 & k_1 a_1 + k_2 a_2 \end{bmatrix} \in U \text{ since}$$

$(k_1 a_1 + k_2 a_2), (k_1 b_1 + k_2 b_2), (k_1 c_1 + k_2 c_2) \in \mathbb{R}$. Therefore, U is a vector subspace of V .

The general element of U can be written as $a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $a, b, c \in \mathbb{R}$,

therefore $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a spanning set for U , which is obviously linearly independent and so is a basis.

Similarly, W is a vector subspace of V with basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$; the proof is almost identical to that for U .

The vector subspace $U \cap W$ is the set of all matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a = -a \Rightarrow a = 0$, i.e. $U \cap W = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \mid b, c \in \mathbb{R} \right\}$, the obvious basis for which is $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

The obvious basis for $U + W$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ (the union of the bases for U and W).

Therefore $\dim(U) = 3$, $\dim(W) = 3$, $\dim(U \cap W) = 2$, $\dim(U + W) = 4$, which satisfies the dimension theorem

$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$ since $4 = 3 + 3 - 2$.